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# Asymptotic scaling describing signal propagation in complex networks

Peng Ji [1,2,3] ✉, Wei Lin [1,2,3,4] and Jürgen Kurths [5,6,7]

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The complex dynamics emergent in diverse systems are influenced not only by the network topologies but also by the interplay of these topologies with the dynamical mechanisms of interaction between nodes. Hens et al.[1] established a general framework, novelly linking the universal topological features to the spatiotemporal propagation of signals in complex networks and analytically capturing the topological role in predicting local responses through the asymptotic scaling relationship. Although using an appropriate form of the asymptotic scaling can reveal universal characteristics in complex systems, it is likely to lead to biased or even incorrect predictions if the scaling form is not accurately estimated. It is possible, however, to achieve substantial improvements in the predictive power by including a suitable prefactor in the scaling.

Hens et al.[1] linked $\tau_i$, the response time of the $i$th node, to the weighted degree of node $i$ $S_i = \sum_{j=1}^{N} A_{ij}$ through the universal scaling relationship with $\tau_i \sim S_i^{\theta}$. Here, as used in the reply by Hens et al.[2], the notation '∼' stands for an asymptotic scaling where the prefactor is not taken into account. Based on this asymptotic scaling, they reported three distinctive dynamic regimes. We have carefully checked the analytical argument presented in ref. [1] and its Supplementary Information, and found that the argument, Supplementary equations (1.32)–(1.42) for analytically approximating the universal exponent, $\theta$, was not mathematically correct. Although, for some special cases indicated below, this incorrect argument can lead to a correct expression of the exponent, it is likely to obtain an incorrect expression resulting in an inaccurate or even false scaling relationship for some other representative cases. After our corrections, we carried out numerical simulations to support our analytical findings.

More precisely, the right-hand side of Supplementary equation (1.32) contains the term $\langle M_2(x)\rangle_{\odot}$, which should have been omitted there. We thus correct this flaw in the following manner:

$$SM_1'(x(S))\langle M_2(x)\rangle_{\odot} = \frac{M_1'(R^{-1}(\lambda))}{\lambda}.$$

More importantly, we found an error in the right-hand side of Supplementary equation (1.36), where $M_1(x)$ appeared in the derivative with respect to the argument $x$. In fact, $M_1(x)$ should be located in front of the derivative. We thus correct the scaling relationship in Supplementary equation (1.36) to

$$\frac{1}{\tau(S)} = -\frac{1}{\lambda^2}\cdot M_1(x)\cdot \left.\frac{dR(x)}{dx}\right|_{x=R^{-1}(\lambda)}. \qquad (1)$$

The original results do not change for some cases, for example, when the function $M_1(x)$ is independent of $x$ in the above correction, which applies for the biological models used in ref. [1] for regulatory dynamics ($\mathbb{R}$) and population dynamics ($\mathbb{P}$). However, using the above correction, the exact expression of $[\tau(S)]^{-1}$ does change for the function $M_1(x)$ explicitly containing $x$, which applies to the models of epidemic spreading ($\mathbb{E}$) and mutualistic dynamics ($\mathbb{M}$). We display the unchanged and/or changed expressions in Table 1 for all these models.

Using the expressions in Table 1, the universal exponent for model $\mathbb{E}$ is the same as the original exponent. However, the changed expression for model $\mathbb{M}$ can impact the value of the universal exponent. To illustrate this impact, we use the Epinions network for model $\mathbb{M}$, following the parameter configurations in ref. [1] but

**Table 1 | Expressions of $[\tau(S)]^{-1}$ using different formulas, where $\mathcal{H}(x) = \frac{x^h}{1+x^h}$ and $F(x) = \frac{\alpha x}{1+\alpha x}$**

| Models | $[\tau(S)]^{-1}$ | $[\tau(S)]^{-1}$ |
|---|---|---|
| | **Using the uncorrected formula** | **Using the corrected formula** |
| $\mathbb{R}$: $\dot{x}_i = -Bx_i^a(t) + \sum_{j=1}^{N} A_{ij}\mathcal{H}(x_j(t))$ | $aB^{\frac{1}{a}}\left[S\langle M_2(x)\rangle_{\odot}\right]^{\frac{a-1}{a}}$ | $aB^{\frac{1}{a}}\left[S\langle M_2(x)\rangle_{\odot}\right]^{\frac{a-1}{a}}$ |
| $\mathbb{P}$: $\dot{x}_i = -Bx_i^b(t) + \sum_{j=1}^{N} A_{ij}x_j^a(t)$ | $bB^{\frac{1}{b}}\left[S\langle M_2(x)\rangle_{\odot}\right]^{\frac{b-1}{b}}$ | $bB^{\frac{1}{b}}\left[S\langle M_2(x)\rangle_{\odot}\right]^{\frac{b-1}{b}}$ |
| $\mathbb{E}$: $\dot{x}_i = -Bx_i(t) + \sum_{j=1}^{N} A_{ij}[1 - x_i(t)]x_j(t)$ | $2S\langle M_2(x)\rangle_{\odot} + B$ | $S\langle M_2(x)\rangle_{\odot} + B$ |
| $\mathbb{M}$: $\dot{x}_i = Bx_i(t)\left[1 - \frac{x_i^a(t)}{C}\right] + \sum_{j=1}^{N} A_{ij}x_i(t)F(x_j(t))$ | $(a-1)S\langle M_2(x)\rangle_{\odot} + aB$ | $aS\langle M_2(x)\rangle_{\odot} + aB$ |

[1]Institute of Science and Technology for Brain-inspired Intelligence and LCNBI, Fudan University, Shanghai, China. [2]State Key Laboratory of Medical Neurobiology and MOE Frontiers Center for Brain Science, Institutes of Brain Science, Fudan University, Shanghai, China. [3]Research Institute of Intelligent Complex Systems, Fudan University, Shanghai, China. [4]School of Mathematical Sciences, SCMS and SCAM, Fudan University, Shanghai, China. [5]Potsdam Institute for Climate Impact Research (PIK), Potsdam, Germany. [6]Nizhny Novgorod State University, Nizhny Novgorod, Russia. [7]Saratov State University, Saratov, Russia. ✉e-mail: pengji@fudan.edu.cn

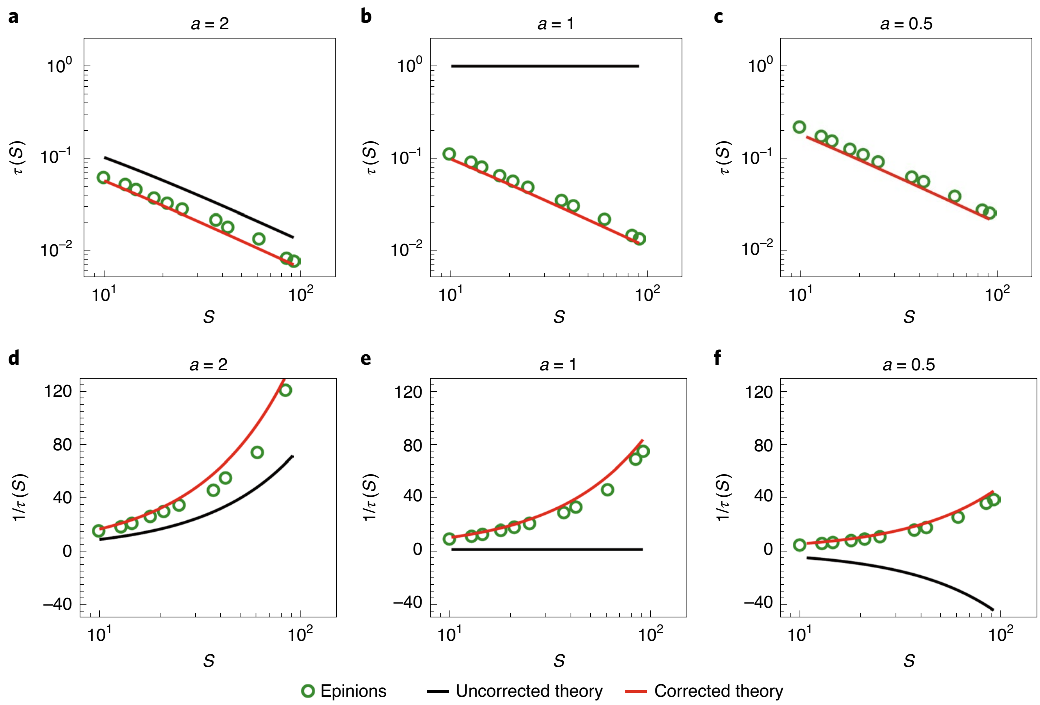


**Fig. 1 | Dynamic regimes of signal propagation for model *M* with the Epinions network.** We evaluate the local response times $\tau(S)$ of all nodes versus their weighted degree $S$ for the mutualistic dynamics. **a–c**, $\tau(S)$ versus $S$ with $a = 2$ (**a**), $a = 1$ (**b**) and $a = 0.5$ (**c**). The green data points represent logarithmic values of $\tau(S)$ from the simulations. The black lines represent the analytical prediction using the uncorrected formula in ref. [1], and the red lines show the prediction using the corrected formula (equation (1)). **d–f**, $1/\tau(S)$ versus $S$ with $a = 2$ (**d**), $a = 1$ (**e**) and $a = 0.5$ (**f**). Here, the green data points represent $1/\tau$ values from the simulations.

only altering the value of the parameter $a$ in the set {2, 1, 0.5}. As shown in Fig. 1, for all three cases of $a$, our analytical prediction using the corrected formula, equation (1), is closely consistent with the numerical simulations, different from the prediction using the uncorrected formula. In particular, from Fig. 1a,d, it is seen that when $a = 2$, both predictions are qualitatively acceptable, while our prediction works more accurately. However, for $a \le 1$, the two predictions are diametrically opposite. For $a = 1$, that is, the classical mutualistic dynamics, the analytical prediction using the original formula implies that the universal exponent $\theta = 0$ and $\tau(S)$ are thus independent of the degree $S$ (Fig. 1b,e). For $a = 0.5$, the prediction using the original formula gives $\tau(S) < 0$ for large $S$, so the system is located in the unphysical regime (Fig. 1c,f). These two cases indicate that the prediction, using the uncorrected formula for the universal exponent, deviates considerably from the exact result, while our corrected formula makes an accurate prediction.

Additionally, to guarantee the analytical soundness of the above results, we need an assumption that the steady state achieved should not coincide with the singularities of the derivative, $M_0'(x)$.

## Online content

Any methods, additional references, Nature Research reporting summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions and competing interests; and statements of data and code availability are available at https://doi.org/10.1038/s41567-020-1025-3.

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## Author contributions
P.J. designed the research and contributed to the modelling. P.J., W.L. and J.K. contributed to the discussion and writing the paper.

## Competing interests
The authors declare no competing interests.

## Additional information
**Correspondence and requests for materials** should be addressed to P.J.